

# THE CLIFFORD TWIST

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ABSTRACT. This is an elementary exposition of the twisted group algebra representation of simple Clifford algebras.

## 1. CLIFFORD ALGEBRA

Clifford Algebra is an algebra defined on a potentially infinite set  $e_1, e_2, e_3, \dots$  of linearly independent unit vectors, their finite products (called multi-vectors) and the unit scalar 1 (denoted  $e_0$ ). Every element of the algebra is a linear combination of these basis elements over some ring, usually the real numbers.

The vectors  $e_1, e_2, e_3, \dots$  are referred to as ‘1-blades.’ A product of two vectors is called a ‘2-blade.’ three vectors a ‘3-blade’ and so forth. The scalar  $e_0$  is a ‘0-blade.’ An  $n$ -blade multi-vector is said to be of grade  $n$ .

There are four fundamental multiplication properties of 1-blades.

- (1) The square of 1-blades is  $\mu$  (where  $\mu^2 = 1$ ).
- (2) The product of 1-blades is anti-commutative.
- (3) The product of 1-blades is associative.
- (4) Every  $n$ -blade can be factored into the product of  $n$  distinct 1-blades.

The product of  $e_i$  and  $e_j$  is denoted  $e_{ij}$  if  $i < j$  and by  $-e_{ij}$  if  $i > j$ . Likewise for higher order blades. For example, if  $i < j < k$  then  $e_i e_j e_k = e_{ijk}$ .

Any two  $n$ -blades may be multiplied by first factoring them into 1-blades. For example, the product of  $e_{134}$  and  $e_{23}$ , is computed as follows:

$$\begin{aligned} e_{134}e_{23} &= e_1e_3e_4e_2e_3 \\ &= -e_1e_4e_3e_2e_3 \\ &= e_1e_4e_2e_3e_3 \\ &= \mu e_1e_4e_2 \\ &= -\mu e_1e_2e_4 \\ &= -\mu e_{124} \end{aligned}$$

## 2. REPRESENTING CLIFFORD ALGEBRA AS A TWISTED GROUP ALGEBRA

Each of the basis elements of Clifford algebra  $1, e_1, e_2, e_{12}, e_3, \dots$  can be associated with an element of the set  $G$  of non-negative integers.

Each vector  $e_k$  is associated with the integer  $2^{k-1}$  and the scalar  $e_0$  is associated with 0. A multi-vector is associated with the sum of the integers associated with its vector factors. Thus, for example, the multi-vector  $e_{134}$  is associated with the sum  $2^0 + 2^2 + 2^3 = 13$ . Notice that the binary representation of 13 is 1101 with

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bits 1, 3 and 4 set. We will represent the sequence  $1, e_1, e_2, e_{12}, e_3, \dots$  by the sequence  $i_0, i_1, i_2, i_3, i_4, \dots$  where the subscript of  $i$  is the number associated with the corresponding vector or multi-vector.

Notice that, since the square of a vector  $e_k$  is  $\mu$  which is either 1 or  $-1$ , the product of two basis elements  $i_p$  and  $i_q$  will always be either  $i_r$  or  $-i_r$  where  $r$  is the XOR (exclusive or) of the binary representations of integers  $p$  and  $q$ . The set  $G$  of non-negative integers is a group under XOR. For brevity, we will denote the operation  $p$  XOR  $q$  by simple concatenation  $pq$ . Thus there is a function  $\phi$  mapping  $G \times G$  into  $\{-1, 1\}$  such that if  $p, q \in G$  then

$$(2.1) \quad i_p i_q = \phi(p, q) i_{pq}$$

thereby representing Clifford algebra as a twisted group algebra.

Let  $2p$  denote the double of  $p$ . Notice that the vector factors of  $i_{2p}$  are the *successors* of the vector factors of  $i_p$  in the sense that  $e_k$  is a vector factor of  $i_{2p}$  if and only if  $e_{k-1}$  is a vector factor of  $i_p$ . For example,  $i_{13} = e_{134}$  and  $i_{26} = e_{245}$ . This is more intuitive if the subscripts are represented in binary.  $13 = 1101_B$  with bits 1,3 and 4 set, and  $2(13) = 26 = 11010_B$  with bits 2,4 and 5 set. Multiplying by 2 in binary shifts bits to the left and appends a 0 on the right.

The next two lemmas are then immediately obvious.

**Lemma 2.1.**

$$e_1 i_{2p} = i_{2p+1}$$

**Lemma 2.2.**

$$e_1 i_{2p+1} = \mu i_{2p}$$

Let  $\beta(p)$  denote the sum of the bits of  $p$ . Then  $\beta(p)$  is the grade of  $i_p$ . The remaining lemmas follow from the fact that  $i_{2p}$  contains exactly  $\beta(p)$  vector factors and  $e_1$  must be ‘commuted’ with each of them to ‘find its place’ so to speak.

**Lemma 2.3.**

$$i_{2p} e_1 = (-1)^{\beta(p)} i_{2p+1}$$

**Lemma 2.4.**

$$i_{2p+1} e_1 = (-1)^{\beta(p)} \mu i_{2p}$$

**Theorem 2.5.** *There is a twist  $\phi(p, q)$  mapping  $G \times G$  into  $\{-1, 1\}$  such that if  $p, q \in G$ , then  $i_p i_q = \phi(p, q) i_{pq}$ .*

*Proof.* Let  $G_n = \{p \mid 0 \leq p < 2^n\}$  with group operation “bit-wise exclusive or.”

To begin with,  $i_0 i_0 = \phi(0, 0) i_0 = 1$  provided  $\phi(0, 0) = 1$ .

This defines the twist for  $G_0$ .

If  $p$  and  $q$  are in  $G_{n+1}$ , then there are elements  $u$  and  $v$  in  $G_n$  such that one of the following is true:

- (1)  $p = 2u$  and  $q = 2v$
- (2)  $p = 2u$  and  $q = 2v + 1$
- (3)  $p = 2u + 1$  and  $q = 2v$
- (4)  $p = 2u + 1$  and  $q = 2v + 1$

Assume  $\phi$  is defined for  $u, v \in G_n$ , then consider these four cases in order.

- (1)  $p = 2u$  and  $q = 2v$

$$\begin{aligned}
 i_p i_q &= i_{2u} i_{2v} \\
 &= \phi(u, v) i_{2uv} \\
 &= \phi(2u, 2v) i_{(2u)(2v)} \\
 &= \phi(p, q) i_{pq}
 \end{aligned}$$

provided  $\phi(2u, 2v) = \phi(u, v)$ .

- (2)  $p = 2u$  and  $q = 2v + 1$

$$\begin{aligned}
 i_p i_q &= i_{2u} i_{2v+1} \\
 &= i_{2u} e_1 i_{2v} \\
 &= (-1)^{\beta(u)} e_1 i_{2u} i_{2v} \\
 &= (-1)^{\beta(u)} e_1 \phi(2u, 2v) i_{2uv} \\
 &= (-1)^{\beta(u)} \phi(u, v) i_{2uv+1} \\
 &= \phi(2u, 2v+1) i_{2uv+1} \\
 &= \phi(p, q) i_{pq}
 \end{aligned}$$

provided  $\phi(2u, 2v+1) = (-1)^{\beta(u)} \phi(u, v)$ .

- (3)  $p = 2u + 1$  and  $q = 2v$

$$\begin{aligned}
 i_p i_q &= i_{2u+1} i_{2v} \\
 &= e_1 i_{2u} i_{2v} \\
 &= e_1 \phi(u, v) i_{2uv} \\
 &= \phi(u, v) i_{2uv+1} \\
 &= \phi(2u+1, v) i_{2uv+1} \\
 &= \phi(p, q) i_{pq}
 \end{aligned}$$

provided  $\phi(2u+1, 2v) = \phi(u, v)$ .

- (4)  $p = 2u + 1$  and  $q = 2v + 1$

$$\begin{aligned}
 i_p i_q &= i_{2u+1} i_{2v+1} \\
 &= e_1 i_{2u} e_1 i_{2v} \\
 &= (-1)^{\beta(u)} e_1 e_1 i_{2u} i_{2v} \\
 &= (-1)^{\beta(u)} \mu \phi(u, v) i_{2uv} \\
 &= \phi(2u+1, 2v+1) i_{2uv} \\
 &= \phi(p, q) i_{pq}
 \end{aligned}$$

provided  $\phi(2u+1, 2v+1) = (-1)^{\beta(u)} \mu \phi(u, v)$ .

□

**Corollary 2.6.** *Assume  $p, q \in G_n$ . The Clifford algebra twist can be defined recursively as follows:*

- (1)  $\phi(0, 0) = 1$
- (2)  $\phi(2p, 2q) = \phi(2p+1, 2q) = \phi(p, q)$
- (3)  $\phi(2p, 2q+1) = (-1)^{\beta(p)} \phi(p, q)$

$$(4) \quad \phi(2p+1, 2q+1) = (-1)^{\beta(p)} \mu \phi(p, q)$$

Stated another way

$$(2.2) \quad \begin{bmatrix} \phi(2p, 2q) & \phi(2p, 2q+1) \\ \phi(2p+1, 2q) & \phi(2p+1, 2q+1) \end{bmatrix} = \phi(p, q) \begin{bmatrix} 1 & (-1)^{\beta(p)} \\ 1 & (-1)^{\beta(p)} \mu \end{bmatrix}$$

### 3. RECURSIVE GENERATION OF TWIST MATRICES FOR HIGHER DIMENSIONS

The twist matrix for dimension one is found when  $p = q = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & \mu \end{bmatrix}$$

For two dimensions,  $0 \leq p \leq 1, 0 \leq q \leq 1$ , the twist matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \mu & 1 & \mu \\ 1 & -1 & \mu & -\mu \\ 1 & -\mu & \mu & -1 \end{bmatrix}$$

For  $\mu = -1$  these coincide with the twist tables for complex numbers and quaternions. For dimension 3, however, we do not get the twist table for the octonions, rather

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mu & 1 & \mu & 1 & \mu & 1 & \mu \\ 1 & -1 & \mu & -\mu & 1 & -1 & \mu & -\mu \\ 1 & -\mu & \mu & -1 & 1 & -\mu & \mu & -1 \\ 1 & -1 & -1 & 1 & \mu & -\mu & -\mu & \mu \\ 1 & -\mu & -1 & \mu & \mu & -1 & -\mu & 1 \\ 1 & 1 & -\mu & -\mu & \mu & \mu & -1 & -1 \\ 1 & \mu & -\mu & -1 & \mu & 1 & -1 & -\mu \end{bmatrix}$$

For dimension four the twist matrix is too large to represent in this form, so we make the following substitutions:

$$(3.1) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & \mu \end{bmatrix}$$

$$(3.2) \quad B = \begin{bmatrix} 1 & -1 \\ 1 & -\mu \end{bmatrix}$$

The matrices  $A$  and  $B$  are simply the values of  $M(p) = \begin{bmatrix} 1 & (-1)^{\beta(p)} \\ 1 & (-1)^{\beta(p)} \mu \end{bmatrix}$  when  $(-1)^{\beta(p)}$  is positive and negative, respectively.

Then the dimension 4 twist table can be represented as follows.

$$\begin{bmatrix} A & A & A & A & A & A & A & A \\ B & \mu B & B & \mu B & B & \mu B & B & \mu B \\ B & -B & \mu B & -\mu B & B & -B & \mu B & -\mu B \\ A & -\mu A & \mu A & -A & A & -\mu A & \mu A & -A \\ B & -B & -B & B & \mu B & -\mu B & -\mu B & \mu B \\ A & -\mu A & -A & \mu A & \mu A & -A & -\mu A & A \\ A & A & -\mu A & -\mu A & \mu A & \mu A & -A & -A \\ B & \mu B & -\mu B & -B & \mu B & B & -B & -\mu B \end{bmatrix}$$

The twist tables for the various dimensions can be generated recursively beginning with  $A$  for dimension 1, then making the following replacements to generate the twist table for each successively higher dimension:

$$(3.3) \quad A \Rightarrow \begin{bmatrix} A & A \\ B & \mu B \end{bmatrix}$$

$$(3.4) \quad B \Rightarrow \begin{bmatrix} B & -B \\ A & -\mu A \end{bmatrix}$$

#### 4. A TREE FOR COMPUTING THE CLIFFORD TWIST

In [3] a tree for computing the Cayley-Dickson twist is described. The same procedure applies to the Clifford twist.

The tree consists of only four components which repeat indefinitely, beginning at node  $A$ . There are two versions, one for each value of  $\mu$ .

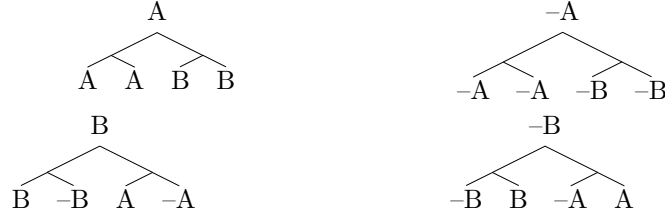


FIGURE 1. Clifford twist tree for  $\mu = 1$ .

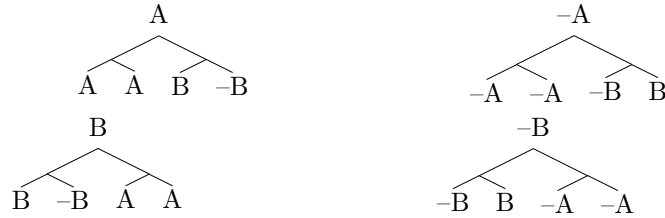


FIGURE 2. Clifford twist tree for  $\mu = -1$ .

Let us illustrate the use of the tree to compute the product  $i_{2636}i_{1143}$  given  $\mu = -1$ .

- (1) Convert the subscripts to binary notation.  $2636 = 101001001100_B$  and  $1143 = 10001110111_B$ .
- (2) Pair the bits of the first subscript with the bits of the second by placing one over the other. Pad the smaller with zero bits if necessary.  

$$\begin{array}{cccccccccccc} \frac{1}{0} \rightarrow, & \frac{0}{1} \rightarrow, & \frac{1}{0} \rightarrow, & \frac{0}{0} \rightarrow, & \frac{0}{0} \rightarrow, & \frac{1}{1} \rightarrow, & \frac{0}{1} \rightarrow, & \frac{0}{1} \rightarrow, & \frac{1}{0} \rightarrow, & \frac{1}{1} \rightarrow, & \frac{0}{1} \rightarrow, & \frac{0}{1} \rightarrow \end{array}$$
- (3) Each binary pair is an instruction for traversing one of the four tree components. A zero is an instruction to move down a left branch and a one is an instruction to move down a right branch. The result is the following path.

$$\begin{array}{ccc} A & \frac{1}{0} \rightarrow & B \\ & \frac{0}{1} \rightarrow & -B \\ & \frac{1}{0} \rightarrow & -A \\ & \frac{0}{0} \rightarrow & -A \\ & \frac{0}{0} \rightarrow & -A \\ & \frac{1}{1} \rightarrow & B \\ & \frac{0}{1} \rightarrow & -B \\ & \frac{0}{1} \rightarrow & B \\ & \frac{1}{0} \rightarrow & A \\ & \frac{1}{1} \rightarrow & -B \\ & \frac{0}{1} \rightarrow & B \\ & \frac{0}{1} \rightarrow & -B \end{array}$$

Since the result is  $-B$ ,  $\phi(2636, 1143) = -1$ . Whenever the result is  $-A$  or  $-B$ ,  $\phi = -1$  and whenever the result is  $A$  or  $B$ ,  $\phi = +1$ . Since  $101001001100 \text{ XOR } 010001110111 = 111000111011 = 3643$  the result is

$$i_{2636} \cdot i_{1143} = -i_{3643}$$

or

$$e_{347ac} \cdot e_{123567b} = -e_{12456abc}$$

#### REFERENCES

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